Analytical expressions for the kinetic decoupling of WIMPs

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Kinetic decoupling of WIMPs: analytic expressions

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We present a general expression for the values of the average kinetic energy and of the temperature of kinetic decoupling of a WIMP, valid for any cosmological model. We show an example of the usage of our solution when the Hubble rate has a power-law dependence on temperature, and we show results for the specific cases of kination cosmology and low-temperature reheating cosmology.

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WIMP freeze-out

Regulated by Boltzmann equation

\[
\frac{d n_\chi}{dt} + 3H n_\chi = -\langle \sigma v \rangle \left( n_\chi^2 - n_{\chi\text{ eq}}^2 \right)
\]

\(\langle \sigma v \rangle\) thermal average: \(\chi \chi \rightarrow \text{SM SM}\)

“Late-time” behavior

\(n_\chi \approx n_{\chi\text{ eq}}, \text{ so } a^3 n_\chi \approx \text{constant}\)

Present DM density

\[
\Omega_\chi h^2 \approx \frac{3 \times 10^{-27} \text{ cm}^3/\text{s}}{\langle \sigma v \rangle}
\]

The “WIMP miracle”: weak interactions

\(\Omega_\chi h^2 = O \left(10^{-1}\right)\)
**Freeze-out vs. kinetic decoupling**

**WIMP annihilation into SM particles**
- Annihilation rate $\Gamma_{\text{ann}}$
- Changes $n_\chi$
- Process ends at
  $$\Gamma_{\text{ann}} \approx H(T_{\text{freeze}})$$
  with $T_{\text{freeze}} \approx M_\chi/20$

**Scattering of WIMP and SM particles**
- Momentum exchange rate $\Gamma_{\text{exc}}$
- Does not change $n_\chi$
- Process ends at
  $$\Gamma_{\text{exc}} \approx H(T_{kd})$$
  with $T_{kd} \ll T_{\text{freeze}}$
Temperature of kinetic decoupling

- $T_{kd}$ is a key parameter in cosmological models.
- determines the cutoff in the density spectrum
- links the size of the smallest halos with WIMP nature

The mass of the smallest halo $M_{cut}$ is the largest between:

- from WIMP free-streaming $M_{fs} = \frac{4\pi}{3} \rho_\chi \left( \frac{\pi}{k_{fs}} \right)^3$
- from acoustic oscillations $M_{ao} = \frac{4\pi}{3} \rho_\chi \left( \frac{1}{H(T_{kd})} \right)^3$
More on kinetic decoupling

The scattering of WIMPs of mass $M_\chi$ off a plasma at temperature $T \ll M_\chi$ is a Brownian motion (random walk)

$$\sqrt{N_e} \Delta p = p \approx \sqrt{M_\chi T}$$

- $p$ : WIMP momentum
- $\Delta p$ : WIMP momentum spread
- $N_e$ : number of collisions required to change the momentum by $p$. 
The WIMP occupation number $f_\chi = f_\chi(p)$ follows a Fokker-Planck (FP) equation

\[ \frac{\partial f_\chi}{\partial t} - H(T) \mathbf{p} \cdot \frac{\partial f_\chi}{\partial \mathbf{p}} = \gamma(T) \frac{\partial}{\partial \mathbf{p}} \left[ \mathbf{p} f_\chi (1 + f_\chi) + M_\chi T \frac{\partial f_\chi}{\partial \mathbf{p}} \right] \]

We solve FP for any $H(T)$ and $\gamma(T)$, assuming $f_\chi \ll 1$
For $f_\chi \ll 1$, we rewrite FP as the Master Equation

$$a \frac{dT_\chi}{da} + 2 [1 + \Upsilon(T)] T_\chi = 2\Upsilon(T) T$$

where we defined the kinetic temperature

$$T_\chi = \frac{2}{3} \int \frac{p_\chi^2}{2M_\chi} f_\chi(p_\chi) d^3p_\chi$$

WIMP scattering off the plasma is regulated by the ratio

$$\Upsilon(T) = \frac{\gamma(T)}{H(T)}$$
The general solution to the Master Equation is the sum of a **homogeneous** and an **inhomogeneous** part

\[
T_X(a) = T_i \left( \frac{a_i}{a} \right)^2 e^{-G(a,a_i)} +
\]

\[
+ \frac{2}{a^2} \int_{a_i}^{a} e^{-G(a,a')} \gamma(a') T(a') a' \, da'
\]

where we introduced \( T_i = T_X(a_i) \) and

\[
G(a, a') = 2 \int_{a'}^{a} \gamma(a'') \frac{da''}{a''} \quad \text{or} \quad G(t, t') = 2 \int_{t'}^{t} \gamma(t'') \, dt''
\]
Limits at early- and late-time

• $\gamma(T) \gg H(T)$ or $\Upsilon(T) \gg 1$:
  
  WIMPs are tightly coupled to the plasma
  
  $$a T_\chi = \text{constant} \quad \text{and} \quad T_\chi \approx T$$

• $\gamma(T) \ll H(T)$ or $\Upsilon(T) \ll 1$:
  
  WIMPs decouple from the plasma
  
  $$a^2 T_\chi = \text{constant} \quad \text{and} \quad T_\chi \propto \frac{T^2}{T_i}$$
Applications: power-law cosmology (1)

We choose the parametrization for the cosmology:

\[ H(T) = H_i \left( \frac{T}{T_i} \right)^\nu \]

\[ a^\alpha T = \text{constant} \]

We also set

\[ \gamma(T) = \gamma_i \left( \frac{T}{T_i} \right)^{4+n} \quad \text{and} \quad \gamma_i = \frac{\gamma_i}{H_i} \]
The solution to the Master Equation is

\[ T^\chi = T^s \lambda^s e^s \left[ \Gamma(1 - \lambda, s) + \lambda \Gamma(-\lambda, s_i) \right] \]

with

\[ s = \frac{2 \gamma}{\alpha (4 + n - \nu)} = \frac{2}{\alpha (4 + n - \nu)} \gamma_i \left( \frac{T}{T_i} \right)^{4+n-\nu} \]

If WIMPs are initially tightly coupled to the plasma,

\[ T^\chi = T^s \lambda^s e^s \Gamma(1 - \lambda, s) \quad (s_i \rightarrow +\infty) \]

For \( \alpha = 1, \nu = 2 \) (RD), and for \( n = 2 \) (p-wave)

\[ T^\chi = T s^{1/4} e^s \Gamma \left( \frac{3}{4}, s \right) \quad \text{(Berschinger, PRD 74 2006)} \]
Temperature of kinetic decoupling (I)

We define \( \gamma(T_{kd}) = H(T_{kd}) \)

In power-law models \( T_{kd} = T_i \gamma_i^{-\frac{1}{4+n-n}} \)

Standard RD: \( T_{kd,\text{std}} = T_i \left( \frac{H(T_i)}{\gamma_i} \right)^{-\frac{1}{2+n}} \)
Temperature of kinetic decoupling (II)

The late-time behavior of $T_\chi$ during RD is

$$T_\chi = \frac{T^2}{T_i} \left( \frac{2 \gamma_i}{2 + n} \right)^{\frac{1}{2+n}} \Gamma \left( \frac{1 + n}{2 + n} \right)$$

Compare with (Bringmann and Hofmann, JCAP 0407, 016 2007)

$$T_\chi = \frac{T^2}{T_{kd, std}} \left( \frac{2}{2 + n} \right)^{\frac{1}{2+n}} \Gamma \left( \frac{1 + n}{2 + n} \right)$$

gives $T_{kd} = \left( \frac{T_{kd, std}}{T_i^{\nu - 2}} \right)^{\frac{1}{4 + n - \nu}}$

$$= \frac{T^2_{kd, std}}{T_{\text{RH}}} \quad \text{for } \nu = 4, n = 2$$

Low – Temperature Reheating model

Gelmini and Gondolo, JCAP 0810, 002 2008

$T_{kd} = T_{kd, std}$ for $\nu = 2$ (RD)
Summary and conclusions

- $T_{kd}$ is a key parameter in cosmological models, since it sets the size of the smallest protohalos.
- $T_{kd}$ is related to the kinetic temperature $T_\chi$ through the Fokker-Planck equation;
- The Fokker-Planck equation is solved in terms of the kinetic temperature $T_\chi$;
- The solution is in terms of $T$, and can be implemented in numerical models like DarkSUSY;
- In power-law models (RD, LTR, Kination), generic expression for $T_\chi$ and $T_{kd}$ are obtained.