Polytropic and Chaplygin $f(T)$-gravity models

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We reconstruct the different $f(T)$-gravity models corresponding to a set of dark energy scenarios containing the polytropic, the standard Chaplygin, the generalized Chaplygin and the modified Chaplygin gas models. We also derive the equation of state parameter of the selected $f(T)$-gravity models and obtain the necessary conditions for crossing the phantom-divide line.

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I. THE $F(T)$ THEORY OF GRAVITY

In the framework of $f(T)$ theory, the action of teleparallel gravity is given by [1–7]

$$I = \frac{1}{2k^2}\int \mathrm{d}^4x\ e\ \Big[f(T) + L_m\Big], \tag{1}$$

where $k^2 = 8\pi G$ and $e = \det(e^i_\mu) = \sqrt{-g}$. Also T and L_m are the torsion scaler and the Lagrangian density of the matter inside the universe, respectively. Note that e^i_μ is the vierbein field which uses as dynamical object in teleparallel gravity and has the following orthonormal property [3]

$$\mathbf{e}_i \cdot \mathbf{e}_j = \eta_{ij}, \tag{2}$$

where $\eta_{ij} = \mathrm{diag}(1, -1, -1, -1)$. Each vector $\mathbf{e}_i$ can be described by its components e^μ_i, where $i = 0, 1, 2, 3$ refers to the tangent space of the manifold and $\mu = 0, 1, 2, 3$ labels coordinates on the manifold. The metric tensor is obtained from the dual vierbein as

$$g_{\mu\nu}(x) = \eta_{ij}e^i_\mu(x)e^j_\nu(x). \tag{3}$$

The torsion scalar T is defined as [3]

$$T = S_\rho{}^{\mu\nu}T^\rho{}_{\mu\nu}, \tag{4}$$

where the non-null torsion tensor $T^\rho{}_{\mu\nu}$ is given by

$$T^\rho{}_{\mu\nu} = e^\rho_i(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu), \tag{5}$$

and

$$S_\rho{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho T^{\alpha\mu}{}_\alpha). \tag{6}$$

Also $K^{\mu\nu}{}_\rho$ is the contorsion tensor and defined as

$$K^{\mu\nu}{}_\rho = -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}). \tag{7}$$

Taking the variation of the action (1) with respect to the vierbein, one can obtain the field equations as [3]

$$S_i{}^{\mu\nu}\partial_\mu(T)f_{TT}(T) + [e^{-1}\partial_\mu(eS_i{}^{\mu\nu}) - e^\lambda_i T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}]f_T(T) + \frac{1}{4}e^\nu_i f(T) = \frac{k^2}{2}e_i{}^\rho T_\rho{}^\nu, \tag{8}$$

where subscript T denotes a derivative with respect to T, $S_i{}^{\mu\nu} = e_i{}^\rho S_\rho{}^{\mu\nu}$ and $T_{\mu\nu}$ is the matter energy-momentum tensor. The set of equations (8) are 2nd order which makes them simpler than the corresponding field equations resulting in the other modified gravity theories like $f(R)$, $f(\mathcal{G})$ and so on [4].

Now if we consider the spatially-flat FRW metric for the universe as

$$g_{\mu\nu} = \mathrm{diag}(-1, a^2(t), a^2(t), a^2(t)), \tag{9}$$

where a is the scale factor, then from Eq. (3) one can obtain

$$e^i_\mu = \mathrm{diag}(1, a(t), a(t), a(t)). \tag{10}$$

Substituting the vierbein (10) into (4) yields [3]

$$T = -6H^2, \tag{11}$$

where $H = \dot{a}/a$ is the Hubble parameter.

Taking $T^\mu_\nu = \mathrm{diag}(-\rho, p, p, p)$ for the matter energy-momentum tensor in the prefect fluid form and using the vierbein (10), then the set of field equations (8) for $i = 0 = \nu$ reduce to [3]

$$12H^2 f_T(T) + f(T) = 2k^2\rho, \tag{12}$$

and for $i = 1 = \nu$ yield

$$48H^2\dot{H}f_{TT}(T) - (12H^2 + 4\dot{H})f_T(T) - f(T) = 2k^2 p. \tag{13}$$

Here ρ and p are the total energy density and pressure of the matter inside the universe, respectively, and satisfy the conservation equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{14}$$

Note that Eqs. (12) and (13) are the modified Friedmann equations in the framework of $f(T)$-gravity in the flat spatial FRW universe. If $f(T)=T$ then Eqs. (12) and (13) transform to the usual Friedmann equations in general relativity (GR). So one can rewrite Eqs. (12) and (13) as [4]

$$\frac{3}{k^2}H^2=\rho+\rho_T, \tag{15}$$

$$\frac{1}{k^2}(2\dot{H}+3H^2)=-(p+p_T), \tag{16}$$

where

$$\rho_T=\frac{1}{2k^2}(2Tf_T-f-T), \tag{17}$$

$$p_T=-\frac{1}{2k^2}[-8\dot{H}Tf_{TT}+(2T-4\dot{H})f_T-f+4\dot{H}-T], \tag{18}$$

are the energy density and pressure due to the torsion contribution and satisfy the energy conservation law

$$\dot{\rho}_T+3H(\rho_T+p_T)=0. \tag{19}$$

The equation of state (EoS) parameter due to the torsion contribution is defined as

$$\omega_T=\frac{p_T}{\rho_T}=-1+\frac{8\dot{H}Tf_{TT}+4\dot{H}f_T-4\dot{H}}{2Tf_T-f-T}. \tag{20}$$

For a given $a=a(t)$, by the help of Eqs. (17) and (18) one can reconstruct the $f(T)$-gravity according to any dark energy (DE) model given by the EoS $p_T=p_T(\rho_T)$ or $\rho_T=\rho_T(a)$.

Here we assume a pole-like phantom scale factor as [8]

$$a(t)=a_0(t_s-t)^{-h},\quad t\leq t_s,\quad h>0. \tag{21}$$

Using Eqs. (11) and (21) one can obtain

$$\begin{aligned} H&=\tfrac{h}{t_s-t},\\ T&=-\tfrac{6h^2}{(t_s-t)^2},\\ \dot{H}&=-\tfrac{T}{6h}. \end{aligned} \tag{22}$$

From Eqs. (21) and (22) the scale factor a can be rewritten in terms of T as

$$a=a_0\left(-\frac{T}{6h^2}\right)^{\frac{h}{2}}. \tag{23}$$

II. POLYTROPIC $F(T)$-GRAVITY MODEL

Here like [1] we reconstruct the $f(T)$-gravity from the polytropic gas DE model. Following [9], the EoS of the polytropic gas is given by

$$p_\Lambda=K\rho_\Lambda^{1+\frac{1}{n}}, \tag{24}$$

where K is a positive constant and n is the polytropic index. Using Eq. (19) the energy density evolves as

$$\rho_\Lambda=\left(Ba^{\frac{3}{n}}-K\right)^{-n}, \tag{25}$$

where B is a positive integration constant [9].

Replacing Eq. (23) into (25) yields

$$\rho_\Lambda=\left(\alpha T^{\frac{3h}{2n}}-K\right)^{-n}, \tag{26}$$

where

$$\alpha=Ba_0^{\frac{3}{n}}\left(-6h^2\right)^{\frac{-3h}{2n}}. \tag{27}$$

Equating (17) with (26), i.e. $\rho_T=\rho_\Lambda$, we obtain the following differential equation

$$2Tf_T-f-T-2k^2\left(\alpha T^{\frac{3h}{2n}}-K\right)^{-n}=0. \tag{28}$$

Solving Eq. (28) gives

$$\begin{aligned} f(T)&=\beta\,T^{1/2}+T\\ &+(-1)^{1+n}\frac{2k^2}{K^n}\,{}_2F_1\left(-\frac{n}{3h},n;1-\frac{n}{3h};\frac{\alpha}{K}T^{\frac{3h}{2n}}\right), \end{aligned} \tag{29}$$

where ${}_2F_1$ denotes the first hypergeometric function. Replacing Eq. (29) into (20) one can obtain the EoS parameter of torsion contribution as

$$\omega_T=-1-\frac{1}{\frac{K}{\alpha}T^{\frac{-3h}{2n}}-1},\quad h>0. \tag{30}$$

Using Eqs. (11) and (27), the above relation can be rewritten as

$$\omega_T=-1-\frac{1}{\frac{K}{B}\left[a_0\left(\frac{H}{h}\right)^h\right]^{\frac{-3}{n}}-1},\quad h>0. \tag{31}$$

We see that for $\frac{K}{B}\left[a_0\left(\frac{H}{h}\right)^h\right]^{\frac{-3}{n}}>1$, $\omega_T<-1$ which corresponds to a phantom accelerating universe.

III. STANDARD CHAPLYGIN $F(T)$-GRAVITY MODEL

The EoS of the standard Chaplygin gas (SCG) DE is given by [10]

$$p_\Lambda=-\frac{A}{\rho_\Lambda}, \tag{32}$$

where A is a positive constant. Inserting the above EoS into the energy conservation equation (19), leads to a density evolving as [10]

$$\rho_\Lambda = \sqrt{A + \frac{B}{a^6}}, \tag{33}$$

where B is an integration constant.

Inserting Eq. (23) into (33) one can get

$$\rho_\Lambda = \sqrt{A + \alpha T^{-3h}}, \tag{34}$$

where

$$\alpha = B{a_0}^{-6}(-6h^2)^{3h}. \tag{35}$$

Equating (34) with (17) one can obtain

$$2Tf_T - f - T - 2k^2\sqrt{A + \alpha T^{-3h}} = 0. \tag{36}$$

Solving the differential equation (36) yields

$$f(T) = \beta T^{1/2} + T \\ -2k^2 A^{\frac{1}{2}}\ {}_2F_1\left(\frac{1}{6h}, \frac{-1}{2}; 1 + \frac{1}{6h}; -\frac{\alpha}{A}T^{-3h}\right). \tag{37}$$

Replacing Eq. (37) into (20) one can get

$$\omega_T = -1 + \frac{1}{\frac{A}{\alpha}T^{3h} + 1}\ , \quad h > 0. \tag{38}$$

Using Eqs. (11) and (35), the above relation can be rewritten as

$$\omega_T = -1 + \frac{1}{\frac{A}{B}\left[a_0\left(\frac{H}{h}\right)^h\right]^6 + 1}\ , \quad h > 0, \tag{39}$$

which for $B < 0$ and $\frac{A}{|B|}\left[a_0\left(\frac{H}{h}\right)^h\right]^6 > 1$ then ω_T can cross the phantom-divide line.

IV. GENERALIZED CHAPLYGIN $F(T)$-GRAVITY MODEL

The EoS of the Generalized Chaplygin Gas (GCG) DE model is given by [11]

$$p_\Lambda = -\frac{A}{\rho_\Lambda^\alpha}, \tag{40}$$

where α is a constant in the range $0 \le \alpha \le 1$ (the SCG corresponds to the case $\alpha = 1$) and A a positive constant. Using Eq. (19), the GCG energy density evolves as [11]

$$\rho_\Lambda = \left(A + \frac{B}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}}, \tag{41}$$

where B is an integration constant.

Substituting Eq. (23) into (41) one can get

$$\rho_\Lambda = \left(A + \gamma\ T^{\frac{-3}{2}h(1+\alpha)}\right)^{\frac{1}{1+\alpha}}, \tag{42}$$

where

$$\gamma = B{a_0}^{-3(1+\alpha)}\left(-6h^2\right)^{\frac{3}{2}h(1+\alpha)}. \tag{43}$$

Equating (42) with (17) gives

$$2Tf_T - f - T - 2k^2\left(A + \gamma\ T^{\frac{-3}{2}h(1+\alpha)}\right)^{\frac{1}{1+\alpha}} = 0. \tag{44}$$

Solving Eq. (44) yields

$$f(T) = \beta T^{1/2} + T - 2k^2 A^{\frac{1}{1+\alpha}} \times \\ {}_2F_1\left(\frac{1}{3h(1+\alpha)}, \frac{-1}{1+\alpha}; 1 + \frac{1}{3h(1+\alpha)}; -\frac{\gamma}{A}T^{\frac{-3}{2}h(1+\alpha)}\right), \tag{45}$$

Replacing Eq. (45) into (20) gives the EoS parameter as

$$\omega_T = -1 + \frac{1}{\frac{A}{\gamma}T^{\frac{3}{2}h(1+\alpha)} + 1}\ , \quad h > 0, \quad 0 \le \alpha \le 1, \tag{46}$$

Using Eqs. (11) and (43), the above relation can be rewritten as

$$\omega_T = -1 + \frac{1}{\frac{A}{B}\left[a_0\left(\frac{H}{h}\right)^h\right]^{3(1+\alpha)} + 1}, \quad h > 0, \quad 0 \le \alpha \le 1, \tag{47}$$

which for $B < 0$ and $\frac{A}{|B|}\left[a_0\left(\frac{H}{h}\right)^h\right]^{3(1+\alpha)} > 1$ then ω_T can cross the phantom-divide line.

V. MODIFIED CHAPLYGIN $F(T)$-GRAVITY MODEL

The EoS of the modified Chaplygin gas (MCG) DE model is given by [12]

$$p_\Lambda = A\rho_\Lambda - \frac{B}{\rho_\Lambda^\alpha}, \tag{48}$$

where A and B are positive constants and $0 \le \alpha \le 1$. Using Eq. (19), the MCG energy density evolves as [12]

$$\rho_\Lambda = \left(\frac{B}{1+A} + \frac{C}{a^{3(1+\alpha)(1+A)}}\right)^{\frac{1}{1+\alpha}}, \tag{49}$$

where C is an integration constant.

Replacing Eq. (23) into (49) yields

$$\rho_\Lambda = \left(\frac{B}{1+A} + \gamma\ T^{\frac{-3}{2}h(1+\alpha)(1+A)}\right)^{\frac{1}{1+\alpha}}, \tag{50}$$

where

$$\gamma = C{a_0}^{-3(1+\alpha)(1+A)}\left(-6h^2\right)^{\frac{3}{2}h(1+\alpha)(1+A)}. \tag{51}$$

Equating (42) with (17) gives

$$2Tf_T - f - T - 2k^2\left(\frac{B}{1+A} + \gamma\, T^{\frac{-3}{2}h(1+\alpha)(1+A)}\right)^{\frac{1}{1+\alpha}} = 0. \tag{52}$$

Solving Eq. (52) yields

$$f(T) = \beta\, T^{1/2} + T - 2k^2\left(\frac{B}{1+A}\right)^{\frac{1}{1+\alpha}} \times {}_2F_1\left(\frac{1}{X}, \frac{-1}{1+\alpha}; 1+\frac{1}{X}; \frac{-\gamma(1+A)}{B}T^{\frac{-X}{2}}\right), \tag{53}$$

where $X := 3h(1+\alpha)(1+A)$.

Replacing Eq. (53) into (20) one can obtain the EoS parameter of torsion contribution as

$$\omega_T = -1 + \frac{A+1}{\frac{B}{\gamma(1+A)}T^{\frac{X}{2}} + 1}\ , \quad h>0, \quad 0\leq\alpha\leq 1, \tag{54}$$

and using Eqs. (11) and (51), it can be rewritten as

$$\omega_T = -1 + \frac{A+1}{\frac{B}{C(1+A)}\left[a_0\left(\frac{H}{h}\right)^h\right]^{3(1+\alpha)(1+A)} + 1}, \tag{55}$$

which for $C<0$ and $\frac{B}{|C|(1+A)}\left[a_0\left(\frac{H}{h}\right)^h\right]^{3(1+\alpha)(1+A)} > 1$ then ω_T can cross the phantom-divide line.

VI. CONCLUSIONS

Here we considered the polytropic gas, the SCG, the GCG and the MCG models of the DE. We reconstructed the different theories of modified gravity based on the $f(T)$ action in the spatially-flat FRW universe and according to the selected DE models. We also obtained the EoS parameter of the polytropic, standard Chaplygin, generalized Chaplygin and modified Chaplygin $f(T)$-gravity scenarios. We showed that crossing the phantom-divide line can occur when the constant parameters of the models to be chosen properly.

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